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A Distortion Theorem in Several Complex Variables

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We have a distortion theorem using invariants related to the Bergman kernel function in several complex variables

Let $K_D(z, \xi)$ be the Bergman kernel function of a bounded domain D in C^n Then

$$T_D(z, \xi) = \frac{\partial^2}{\partial \xi^* \partial z} \log K_D(z, \xi)$$

is uniquely defined by D when $K_D(z, \xi) \neq 0$ ([1])

Moreover we define

$$K_{D,(p,q)}(z, \xi) = K_D(z, \xi)^p (\det T_D(z, \xi))^q,$$

$$T_{D,(p,q)}(z, \xi) = \frac{\partial^2}{\partial \xi^* \partial z} \log K_{D,(p,q)}(z, \xi) \qquad (p, q \geqq 0)$$

For $p = 1$ and $q = 0$, $K_{D,(p,q)}(z, \xi)$ and $T_{D,(p,q)}(z, \xi)$ mean ordinary $K_D(z, \xi)$ and $T_D(z, \xi)$ respectively Then we have the following transformation law for a biholomorphic mapping f from D onto $\varDelta$

$$(1) \qquad K_{\varDelta,(p,q)}(f(z), f(\xi)) \left(\det \frac{\partial f(z)}{\partial z}\right)^{p+q} \overline{\left(\det \frac{\partial f(\xi)}{\partial \xi}\right)^{p+q}} = K_{D,(p,q)}(z, \xi)$$

, where $\det \frac{\partial f(z)}{\partial z}$ means the Jacobian of f with respect to complex variables $z = (z_1, \quad , z_n)'$ ([2] , [3])

Throughout this paper we use the following notations

$$z = (z_1, \quad , z_n)',$$
$$f = f(z) = (f_1(z), \quad , f_n(z))'$$
$$\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial z_1}, \quad , \frac{\partial}{\partial z_n}\right)$$
$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z} \times f$$

, where the symbols $'$, $*$ and $\times$ stand for transposition, conjugated transposition and Kronecker product, respectively

We have the following

THEOREM 1 *Let D be a bounded domain in C^n and let f be an injective holomorphic mapping from D into itself Then we have*

$$K_D(f(z), f(z))\left|\det \frac{\partial f(z)}{\partial z}\right|^2 \leqq K_D(z, z) \quad \text{for } z \in D$$

Proof Let $G = f(D) \subset D$, then

$$(2) \qquad K_D(f(z), f(\xi)) \leqq K_G(f(z), f(\xi)) \qquad z, \xi \in D$$

Since f is a biholomorphic mapping from D onto G, we have

$$(3) \qquad K_G(f(z), f(\xi))\left(\det \frac{\partial f(z)}{\partial z}\right)\overline{\left(\det \frac{\partial f(\xi)}{\partial \xi}\right)} = K_D(z, \xi)$$

by transformation law (1) as $p = 1$ and $q = 0$
Putting $\xi = z$, from (2) and (3), we obtain

$$K_D(f(z), f(z))\left|\det \frac{\partial f(z)}{\partial z}\right|^2 \leqq K_D(z, z)$$

Q E D

THEOREM 2 *Let D be a homogeneous bounded domain in C^n and let f be a holomorphic mapping from D into itself Then we have*

$$(4) \qquad K_{D,(p,q)}(f(z), f(z))\left|\det \frac{\partial f(z)}{\partial z}\right|^{2(p+q)} \leqq K_{D,(p,q)}(z, z) \quad \text{for } p, q \geqq 0$$

Moreover if the equality sign in (4) *holds for at least one point $z \in D$, then the mapping f is necessarily biholomorphic*

Proof Let $w^0 = f(z^0)$ for $z^0 \in D$ Since D is homogeneous, there exists a biholomorphic mapping φ of D onto itself such that $\varphi(w^0) = z^0$ Then the composition $g = \varphi \circ f$ of two mappings f and φ is a mapping which leaves z^0 invariant By the theorem of Carathéodory-Cartan, we have

$$\left|\left[\det \frac{\partial g(z)}{\partial z}\right]_{z=z^0}\right| \leqq 1$$

and if the equality sign holds, then g is biholomorphic mapping from D onto itself ([4]) On the other hand,

$$\left[\det \frac{\partial g(z)}{\partial z}\right]_{z=z^0} = \left[\det \frac{\partial \varphi(w)}{\partial w}\right]_{w=w^0} \left[\det \frac{\partial f(z)}{\partial z}\right]_{z=z^0}$$

Hence we obtain Theorem 2 by the transformation law (1) Q E D

REMARK When $p=1$ and $q=0$, (4) reduces to

$$K_D(f(z), f(z))\left|\det \frac{\partial f(z)}{\partial z}\right|^2 \leqq K_D(z, z) \qquad ([5])$$

This is a generalization of Schwarz Lemma in terms of the Bergman kernel function

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